Interval Prediction for Continuous-Time Systems with Parametric Uncertainties

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Contents

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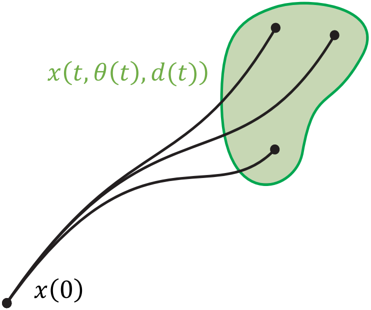
Problem statement

Linear Parameter-Varying systems

$$\dot{x}(t) = A(\theta(t))x(t) + Bd(t)$$

There are two sources of uncertainty:

- Parametric uncertainty $\theta(t)$
- External perturbations $d(t)$



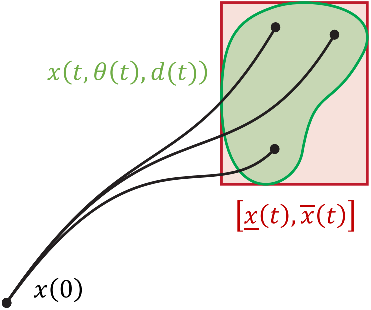
The goal

Interval Prediction

Can we design an interval predictor $[\underline{x}(t), \overline{x}(t)]$ that verifies:

- inclusion property: $\forall t, \underline{x}(t) \leq x(t) \leq \overline{x}(t)$;
- stable dynamics?

We want the predictor to be as tight as possible.



Assumption (Bounded trajectories)

- $\|x\|_\infty < \infty$
- $x(0) \in [\underline{x}_0, \overline{x}_0]$ *for some known* $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$

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Assumption (Bounded parameters)

- $\theta(t) \in \Theta$ *for some known* Θ
- *The matrix function* $A(\theta)$ *is known*

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Assumption (Bounded perturbations)

- $d(t) \in [\underline{d}(t), \overline{d}(t)]$ *for some known signals* $\underline{d}, \overline{d} \in \mathcal{L}_\infty^n$

How to proceed?

A first idea

Assume that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, for some $t \geq 0$.

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↳ Why not use interval arithmetics?

Lemma (Image of an interval)

If A a known matrix, then

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}.$$

where $A^+ = \max(A, 0)$ and $A^- = A - A^+$.

Assume that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, for some $t \geq 0$.

↳ To propagate the interval to $x(t+dt)$, we need to bound $A(\theta(t))x(t)$.

↳ Why not use interval arithmetics?

Lemma (Product of intervals)

If A is unknown but bounded $\underline{A} \leq A \leq \overline{A}$,

$$\underline{A}^{+}\underline{x}^{+} - \overline{A}^{+}\underline{x}^{-} - \underline{A}^{-}\overline{x}^{+} + \overline{A}^{-}\overline{x}^{-} \leq Ax$$

$$\leq \overline{A}^{+}\overline{x}^{+} - \underline{A}^{+}\overline{x}^{-} - \overline{A}^{-}\underline{x}^{+} + \underline{A}^{-}\underline{x}^{-}.$$

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✓ Since $A(\theta)$ and the set Θ are known,
we can easily compute such bounds $\underline{A} \leq A(\theta(t)) \leq \overline{A}$

Following this result, define the predictor:

$$\begin{aligned}
\dot{\underline{x}}(t) &= \underline{A}^{+}\underline{x}^{+}(t) - \overline{A}^{+}\underline{x}^{-}(t) - \underline{A}^{-}\overline{x}^{+}(t) \\
&\quad + \overline{A}^{-}\overline{x}^{-}(t) + B^{+}\underline{d}(t) - B^{-}\overline{d}(t), \\
\dot{\overline{x}}(t) &= \overline{A}^{+}\overline{x}^{+}(t) - \underline{A}^{+}\overline{x}^{-}(t) - \overline{A}^{-}\underline{x}^{+}(t) \\
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&\quad \underline{x}(0) = \underline{x}_0,\ \overline{x}(0) = \overline{x}_0,
\end{aligned} \tag{1}$$

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Proposition (Inclusion property)

✓ *The predictor* (1) *satisfies* $\underline{x}(t) \le x(t) \le \overline{x}(t)(t)$

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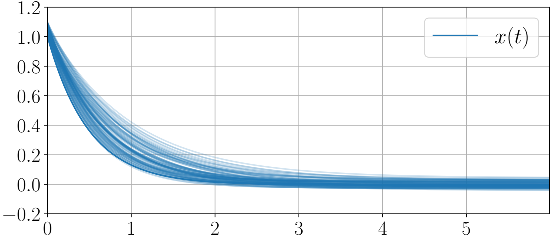
? But is it stable?

Consider the scalar system, for all $t \geq 0$:

$$\dot{x}(t) = -\theta(t)x(t) + d(t), \text{ where } \begin{cases} x(0) \in [\underline{x}_0, \overline{x}_0] = [1.0, 1.1], \\ \theta(t) \in \Theta = [\underline{\theta}, \overline{\theta}] = [1, 2], \\ d(t) \in [\underline{d}, \overline{d}] = [-0.1, 0.1], \end{cases}$$



✓ The system is always stable

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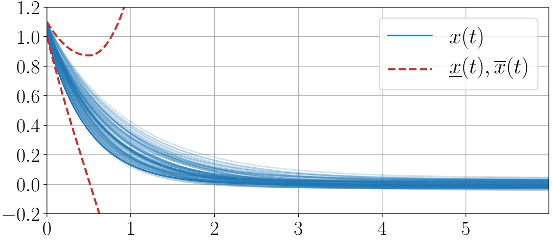

✓ The system is always stable ✗ The predictor (1) is unstable

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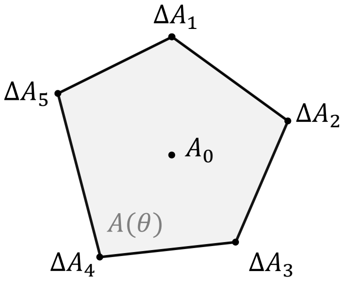
Our proposed predictor

Assumption (Polytopic Structure)

There exist A_0 Metzler and $\Delta A_0, \cdots, \Delta A_N$ such that:

$$A(\theta) = \underbrace{A_0}_{\substack{\textit{Nominal}\\ \textit{dynamics}}} + \sum_{i=1}^{N} \lambda_i(\theta)\Delta A_i, \quad \sum_{i=1}^{N} \underbrace{\lambda_i(\theta)}_{\geq 0} = 1; \quad \forall \theta \in \Theta$$



Denote

$$\Delta A_{+}=\sum_{i=1}^{N}\Delta A_i^{+},\ \Delta A_{-}=\sum_{i=1}^{N}\Delta A_i^{-},$$

We define the predictor

$$\begin{aligned}
\dot{\underline{x}}(t) &= A_0\underline{x}(t)-\Delta A_{+}\underline{x}^{-}(t)-\Delta A_{-}\overline{x}^{+}(t)\\
&\quad +B^{+}\underline{d}(t)-B^{-}\overline{d}(t),\\
\dot{\overline{x}}(t) &= A_0\overline{x}(t)+\Delta A_{+}\overline{x}^{+}(t)+\Delta A_{-}\underline{x}^{-}(t) \qquad (2)\\
&\quad +B^{+}\overline{d}(t)-B^{-}\underline{d}(t),\\
&\quad \underline{x}(0)=\underline{x}_0,\ \overline{x}(0)=\overline{x}_0
\end{aligned}$$

Theorem (Inclusion property)

The predictor (2) ensures $\underline{x}(t)\leq x(t)\leq\overline{x}(t)(t)$.

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Theorem (Stability)

If there exist diagonal matrices P, Q, Q_+, Q_-, Z_+, Z_-, Ψ_+, Ψ_-, Ψ, $\Gamma \in \mathbb{R}^{2n\times 2n}$ *such that the following LMIs are satisfied:*

$$P + \min\{Z_+, Z_-\} > 0,\ \Upsilon \preceq 0,\ \Gamma > 0,$$
$$Q + \min\{Q_+, Q_-\} + 2\min\{\Psi_+, \Psi_-\} > 0,$$

then the predictor (2) *is input-to-state stable with respect to the inputs* $\underline{d}$, $\overline{d}$.

where

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & P \\ \Upsilon_{12}^\top & \Upsilon_{22} & \Upsilon_{23} & Z_+ \\ \Upsilon_{13}^\top & \Upsilon_{23}^\top & \Upsilon_{33} & Z_- \\ P & Z_+ & Z_- & -\Gamma \end{bmatrix},$$

$$\Upsilon_{11} = \mathcal{A}^\top P + P\mathcal{A} + Q,\ \Upsilon_{12} = \mathcal{A}^\top Z_+ + PR_+ + \Psi_+,$$

$$\Upsilon_{13} = \mathcal{A}^\top Z_- + PR_- + \Psi_-,\ \Upsilon_{22} = Z_+R_+ + R_+^\top Z_+ + Q_+,$$

$$\Upsilon_{23} = Z_+R_- + R_+^\top Z_- + \Psi,\ \Upsilon_{33} = Z_-R_- + R_-^\top Z_- + Q_-,$$

$$\mathcal{A} = \begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix},\ R_+ = \begin{bmatrix} 0 & -\Delta A_- \\ 0 & \Delta A_+ \end{bmatrix},\ R_- = \begin{bmatrix} \Delta A_+ & 0 \\ -\Delta A_- & 0 \end{bmatrix},$$

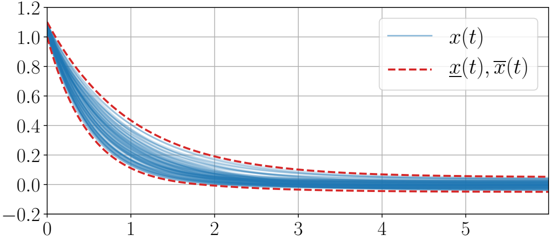
1. Define the extended state vector as $X = [\underline{x}^\top \ \ \overline{x}^\top]^\top$
2. It follows the dynamics
$\dot{X}(t) = \mathcal{A}X(t) + R_+X^+(t) - R_-X^-(t) + \delta(t)$
3. Consider a candidate Lyapunov function:

$$V(X) = X^\top PX + X^\top Z_+X^+ - X^\top Z_-X^-$$

4. $V(X)$ is positive definite provided that $P + \min\{Z_+, Z_-\} > 0$,
5. Check on which condition we have $\dot{V}(X) \leq 0$

Recall the scalar system, for all $t \geq 0$:

$$\dot{x}(t) = -\theta(t)x(t) + d(t), \text{ where } \begin{cases} x(0) \in [\underline{x}_0, \overline{x}_0] = [1.0, 1.1], \\ \theta(t) \in \Theta = [\underline{\theta}, \overline{\theta}] = [1, 2], \\ d(t) \in [\underline{d}, \overline{d}] = [-0.1, 0.1], \end{cases}$$



✓ The system is always stable

✓ The predictor (2) is stable

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Application to autonomous driving

$$\dot{z}_i = f_i(Z, \theta_i),\ i = \overline{1, N},$$

where

- $z_i = [x_i, y_i, v_i, \psi_i]^\top \in \mathbb{R}^4$ is the state of an agent
- $\theta_i \in \mathbb{R}^5$ is the corresponding unknown behavioural parameters
- $Z = [z_1, \dots, z_N]^\top \in \mathbb{R}^{4N}$ is the joint state of the traffic
- $\theta = [\theta_1, \dots, \theta_N]^\top \in \Pi \subset \mathbb{R}^{5N}$

Each vehicle follows the Kinematic Bicycle Model

$$\begin{aligned}
\dot{x}_i &= v_i \cos(\psi_i), \\
\dot{y}_i &= v_i \sin(\psi_i), \\
\dot{v}_i &= a_i, \\
\dot{\psi}_i &= \frac{v_i}{l} tan(\beta_i),
\end{aligned}$$

A linear controller using three features inspired from the intelligent driver model (IDM) [**?**].

$$a_i = \begin{bmatrix}\theta_{i,1} & \theta_{i,2} & \theta_{i,3}\end{bmatrix} \begin{bmatrix} v_0 - v_i \\ -(v_{f_i} - v_i)^- \\ -(x_{f_i} - x_i - (d_0 + v_i T))^- \end{bmatrix},$$

where v_0, d_0 and T respectively denote the speed limit, jam distance and time gap given by traffic rules.

A cascade controller of lateral position y and heading ψ:

$$\begin{aligned}\dot{\psi}_i &= \theta_{i,5}\left(\psi_{L_i} + \sin^{-1}\left(\frac{\widetilde{v}_{i,y}}{v_i}\right) - \psi_i\right), \qquad (3)\\ \widetilde{v}_{i,y} &= \theta_{i,4}(y_{L_i} - y_i).\end{aligned}$$

We assume that the drivers choose their steering command β_i such that (3) is always achieved: $\beta_i = \tan^{-1}(\frac{l}{v_i}\dot{\psi}_i)$.

Linearize trigonometric operators around $y_i = y_{L_i}$ and $\psi_i = \psi_{L_i}$. This yields the following longitudinal dynamics:

$$\dot{x}_i = v_i,$$
$$\dot{v}_i = \theta_{i,1}(v_0 - v_i) + \theta_{i,2}(v_{f_i} - v_i) + \theta_{i,3}(x_{f_i} - x_i - d_0 - v_i T),$$

where $\theta_{i,2}$ and $\theta_{i,3}$ are set to 0 whenever the corresponding features are not active.

$$\dot{Z} = A(\theta)(Z - Z_c) + d.$$

For example, in the case of two vehicles only:

$$Z = \begin{bmatrix} x_i \\ x_{f_i} \\ v_i \\ v_{f_i} \end{bmatrix}, \quad Z_c = \begin{bmatrix} -d_0 - v_0 T \\ 0 \\ v_0 \\ v_0 \end{bmatrix}, \quad d = \begin{bmatrix} v_0 \\ v_0 \\ 0 \\ 0 \end{bmatrix}$$

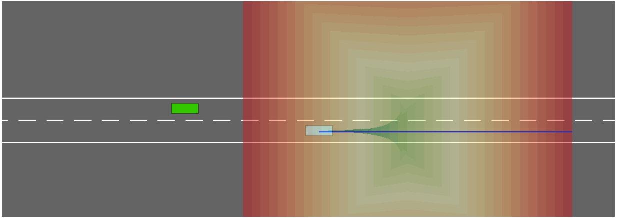
$$A(\theta) = \begin{array}{c} \\ i \\ f_i \\ i \\ f_i \end{array} \begin{array}{c} \begin{array}{cccc} i & f_i & i & f_i \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\theta_{i,3} & \theta_{i,3} & -\theta_{i,1} - \theta_{i,2} - \theta_{i,3} & \theta_{i,2} \\ 0 & 0 & 0 & -\theta_{f_i,1} \end{bmatrix} \end{array}$$

The lateral dynamics are in a similar form:

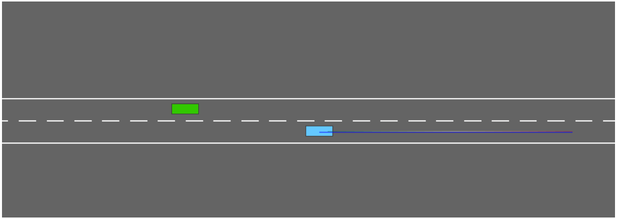
$$\begin{bmatrix} \dot{y}_i \\ \dot{\psi}_i \end{bmatrix} = \begin{bmatrix} 0 & v_i \\ -\frac{\theta_{i,4}\theta_{i,5}}{v_i} & -\theta_{i,5} \end{bmatrix} \begin{bmatrix} y_i - y_{L_i} \\ \psi_i - \psi_{L_i} \end{bmatrix} + \begin{bmatrix} v_i \psi_{L_i} \\ 0 \end{bmatrix}$$

Here, the dependency in v_i is seen as an uncertain parametric dependency, *i.e.* $\theta_{i,6} = v_i$, with constant bounds assumed for v_i using an overset of the longitudinal interval predictor.

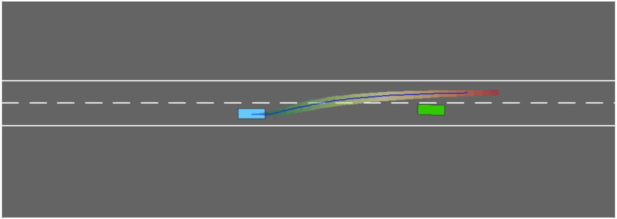
Results



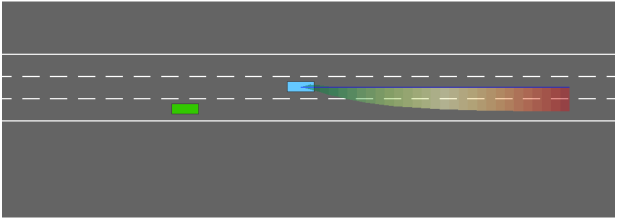
The naive predictor (1) quickly diverges



The proposed predictor (2) remains stable



Prediction during a lane change maneuver



Prediction with uncertainty in the followed lane L_i

Conclusion